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ORIGINAL ARTICLE

A note on fixed point theorems for fuzzy mappings



R. Mohanraj

Department of Mathematics, Regional Centre, Anna University Tirunelveli Region, Tirunelveli 627 007, India

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Abstract In this paper, a common fixed point theorem for contractive type fuzzy mappings in a complete metric space is proved due to Cho (2005) [1]. Further an example is given for the results of Cho (2005) [1, Theorem 3.1] and Park and Jeong (1997) [2, Theorem 3.2] which are not satisfying the condition “for all $x, y \in X$ ” and have a fixed point.

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1. Introduction and preliminaries

Let (X, d) be a metric space. Fixed points for multivalued mapping $T: X \rightarrow 2^X$ are defined as $x \in Tx$ for some $x \in X$. Let $\mathcal{CB}(X)$ denote the set of all nonempty closed and bounded subsets of X . A multivalued mapping $T: X \rightarrow \mathcal{CB}(X)$ is called a contraction mapping if there exists $q \in (0, 1)$ such that

$$H(T(x), T(y)) \leq qd(x, y) \quad \text{for all } x, y \in X,$$

where the Hausdorff metric $H(A, B)$ on $\mathcal{CB}(X)$ is given by

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\},$$

$$\text{where } d(x, C) = \inf_{y \in C} d(x, y)$$

for any nonempty closed and bounded subsets A, B and C of X and for any point $x \in X$.

E-mail address: vmrj@yahoo.com

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A fuzzy set in X is a function with domain X and values in $[0, 1]$. If A is a fuzzy set and $x \in X$, then the function values $A(x)$ are called the grade of membership of x in A . Let $\mathcal{F}(X)$ be the collection of all fuzzy sets on X and let ${}^{\alpha}A = \{x \in X : A(x) \geq \alpha\}$ denote the α -cut of $A \in \mathcal{F}(X)$. The zero-cut of A is defined as the closure of the set $\{x \in X : A(x) > 0\}$.

A mapping F from X to $\mathcal{F}(Y)$ is called a fuzzy mapping if for each $x \in X$, $F(x)$ is a fuzzy set on Y and $F(x)(y)$ denotes the degree of membership of y in $F(x)$. Let X be a metric linear space and let $\mathcal{W}(X)$ denote the set of all fuzzy sets on X such that each of its α -cut is a nonempty compact and convex subset (approximate quantity) of X . A fuzzy mapping F from X to $\mathcal{W}(X)$ is called a fuzzy contraction mapping if there exists $q \in (0, 1)$ such that

$$D(F(x), F(y)) \leq qd(x, y) \quad \text{for each } x, y \in X,$$

$$\text{where } D(A, B) = \sup_{\alpha} H({}^{\alpha}A, {}^{\alpha}B)$$

Define $p_{\alpha}(A, B) = \inf_{x \in {}^{\alpha}A, y \in {}^{\alpha}B} d(x, y)$ and $p(A, B) = \sup_{\alpha} p_{\alpha}(A, B)$ for any fuzzy sets $A, B \in \mathcal{W}(X)$.

It is known that p_{α} is non-decreasing function of α .

Heilpern [3] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction

mappings which is a fuzzy analogue of the fixed point theorem of Nadler [4]. Bose and Sahani [5] extended the Heilpern's result for a pair of generalized fuzzy contraction mappings. Marudai and Srinivasan [6] generalized the Heilpern's result using the Nadler's result. They also obtained a nontrivial generalization of the Nadler's fixed point theorem for fuzzy contraction mappings under weaker settings. Further Vijayaraju and Marudai [7] generalize the result of Bose and Mukerjee [8] for contractive type fuzzy mappings in complete metric spaces. The significance of these results is assuming "each of its α -cut of fuzzy set is nonempty closed and bounded subset of X " instead of approximate quantity of X ". Akbar Azam and Muhammad Arshad [9] proved the result of Vijayaraju and Marudai [7, Theorem 3.1] is incomplete and corrected the proof in right direction. In this paper a common fixed point theorem for contractive type fuzzy mappings in complete metric space due to Cho [1] is proved by using the concept of Vijayaraju and Marudai [7]. Further an example is given for the results of Cho [1, Theorem 3.1] and Park and Jeong [2, Theorem 3.2] which are not satisfying the condition "for all $x, y \in X$ " and have a fixed point.

2. Main results

The following lemma due to Nadler [4] is the main key of our result.

Lemma 2.1 [4]. Let (X, d) be a metric space and $A, B \in CB(X)$, then for each $a \in A, k > 0$ there exists an element $b \in B$ such that $d(a, b) \leq H(A, B) + k$.

Cho [1] and Park and Jeong [2] proved some fixed point theorems for fuzzy mappings from X to $W(X)$ under the contractive type conditions in complete metric space. The following example shows that the condition "for all $x, y \in X$ " fails for the results [1, Theorem 3.1] and [2, Theorem 3.2].

Theorem 2.2 [1]. Let $F, G : X \rightarrow W(X)$ be fuzzy mappings satisfying the following condition: There exists $k \in (0, 1)$ such that

$$D(Fx, Gy) \leq \frac{k}{\sqrt{2}} \{p(x, Fx)p(y, Gy) + p(y, Gy)d(x, y)\}^{\frac{1}{2}} \quad (*)$$

for all $x, y \in X$. Then F and G have a common fixed point.

Theorem 2.3 [2]. Let $F, G : X \rightarrow W(X)$ be fuzzy mappings satisfying the following condition: There exists $k \in (0, 1)$ such that

$$D(Fx, Gy) \leq k\{p(x, Fx)p(y, Gy)\}^{\frac{1}{2}} \quad (**)$$

for all $x, y \in X$. Then F and G have a common fixed point.

Example 2.4. Let $X = [0, 1]$. For $x, y \in X, d(x, y) = |x - y|, \alpha \in (0, 1]$. Define $F, G : X \rightarrow W(X)$ by

$$F(0)(z) = \begin{cases} 1, & z = 0 \\ \frac{1}{2}, & 0 < z \leq 1/50 \\ 0, & z > 1/50 \end{cases} \quad G(0)(z) = \begin{cases} 1, & z = 0 \\ 1/4, & 0 < z \leq 1/100 \\ 0, & z > 1/100 \end{cases}$$

$$F(x)(z) = \begin{cases} \alpha, & 0 \leq z \leq x/25 \\ \frac{z}{2}, & x/25 < z \leq x/10 \\ 0, & z > x/10 \end{cases} \quad G(x)(z) = \begin{cases} \alpha, & 0 \leq z \leq x/20 \\ \frac{z}{2}, & x/20 < z \leq x/10 \\ 0, & z > x/10 \end{cases}$$

Here ${}^1F(x) = {}^1G(x) = \{0\}$ and ${}^2F(x) = [0, x/25]$ and ${}^2G(x) = [0, x/20]$

$$D(F(x), G(y)) = \sup_{\alpha} H({}^{\alpha}F(x), {}^{\alpha}G(y)) = |x/20 - y/25| \\ \leq \frac{k}{\sqrt{2}} [|x - x/25| \cdot |y - y/20| + |y - y/20| |x - y|]^{\frac{1}{2}}$$

For $x = y, F$ and G satisfy all the conditions of Theorem 2.2 and 0 is the common fixed point of F and G .

For $x \neq y$, the condition (*) fails for taking the values $x = 1, y = 0$.

Similarly the condition (**) of Theorem 2.3 fails also.

From the above example, we observe that Theorem 2.2 holds for assuming the condition for all $x \in X$ and for all nonzero values of y in X and Theorem 2.3 holds for assuming the condition for all nonzero values $x, y \in X$.

Next a common fixed theorem for fuzzy mappings is proved due to Cho [1].

Theorem 2.5 [1]. Let $F, G : X \rightarrow W(X)$ be fuzzy mappings satisfying the following condition: There exist $\alpha, \beta > 0$ such that $\alpha + \beta < 1$ and

$$D(Fx, Gy) \leq \frac{\alpha p(y, Gy)[(1 + p(x, Fx))p(x, Fx)]^{\frac{1}{2}}}{1 + 2d(x, y)} + \beta d(x, y),$$

for all $x, y \in X$. Then F and G have a common fixed point.

Theorem 2.6. Let (X, d) be a complete metric space and let F_1 and F_2 be fuzzy mappings from X to $\mathcal{F}(X)$ satisfying the following condition:

- (i) For each $x, y \in X$, there exists $\alpha(x), \alpha(y) \in (0, 1]$ such that ${}^{\alpha(x)}F_1(x)$ and ${}^{\alpha(y)}F_2(y)$ are nonempty closed bounded subsets of X .

- (ii)

$$H({}^{\alpha(x)}F_1(x), {}^{\alpha(y)}F_2(y)) \\ \leq \frac{a_1 d(y, {}^{\alpha(y)}F_2(y)) \{ [1 + d(x, {}^{\alpha(x)}F_1(x))] d(x, {}^{\alpha(x)}F_1(x)) \}^{\frac{1}{2}}}{1 + 2d(x, y)} + a_2 d(x, y),$$

where $a_1, a_2 > 0$ and $a_1 + a_2 < 1$.

Then there exists $z \in X$ such that $z \in {}^{\alpha(z)}F_1(z) \cap {}^{\alpha(z)}F_2(z)$.

Proof. Let $x_0 \in X$. Then by condition (i), there exists $\alpha_1 \in (0, 1]$ such that ${}^{\alpha_1}F_1(x_0)$ is a nonempty closed bounded subset of X .

Choose $x_1 \in {}^{\alpha_1}F_1(x_0)$.

For this x_1 , there exists $\alpha_2 \in (0, 1]$ such that ${}^{\alpha_2}F_2(x_1)$ is a nonempty closed bounded subset of X . Since ${}^{\alpha_1}F_1(x_0)$ and ${}^{\alpha_2}F_2(x_1)$ are nonempty closed bounded subsets of X and by Lemma 2.1, there exists $x_2 \in {}^{\alpha_2}F_2(x_1)$ such that

$$\begin{aligned}
d(x_1, x_2) &\leq H({}^{a_1}F_1(x_0), {}^{a_2}F_2(x_1)) + a_2 \\
&\leq \frac{a_1 d(x_1, {}^{a_2}F_2(x_1)) [1 + d(x_0, {}^{a_1}F_1(x_0))]^{\frac{1}{2}}}{1 + 2d(x_0, x_1)} + a_2 d(x_0, x_1) + a_2 \\
&\leq \frac{a_1 d(x_1, x_2) [1 + d(x_0, x_1)]^{\frac{1}{2}}}{1 + 2d(x_0, x_1)} + a_2 d(x_0, x_1) + a_2 \\
&\leq \frac{a_1 d(x_1, x_2) [2\{1 + d(x_0, x_1)\}d(x_0, x_1)]^{\frac{1}{2}}}{1 + 2d(x_0, x_1)} + a_2 d(x_0, x_1) + a_2 \\
&\leq \frac{a_1 d(x_1, x_2) [1 + 2d(x_0, x_1)]}{1 + 2d(x_0, x_1)} + a_2 d(x_0, x_1) + a_2 \\
&= a_1 d(x_1, x_2) + a_2 d(x_0, x_1) + a_2.
\end{aligned}$$

Therefore $d(x_1, x_2) \leq \frac{a_2}{1-a_1} d(x_0, x_1) + \frac{a_2}{1-a_1} = kd(x_0, x_1) + k$, where $k = \frac{a_2}{1-a_1}$.

For this x_2 , there exists $x_3 \in (0, 1]$ such that ${}^{a_3}F_1(x_2)$ is a nonempty closed bounded subset of X .

Since ${}^{a_3}F_1(x_2)$ and ${}^{a_2}F_2(x_1)$ are nonempty closed bounded subsets of X , there exists $x_3 \in {}^{a_3}F_1(x_2)$ such that

$$d(x_2, x_3) \leq H({}^{a_3}F_1(x_2), {}^{a_2}F_2(x_1)) + ka_2$$

and we obtain

$$\begin{aligned}
d(x_2, x_3) &\leq kd(x_1, x_2) + k^2 \\
&\leq k^2 d(x_0, x_1) + 2k^2.
\end{aligned}$$

Continuing this process there exists sequence $\{x_n\}$ of X such that

$$x_{2n+1} \in {}^{a_{2n+1}}F_1(x_{2n}) \text{ and } x_{2n+2} \in {}^{a_{2n+2}}F_2(x_{2n+1}) \text{ and}$$

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &\leq H({}^{a_{2n+1}}F_1(x_{2n}), {}^{a_{2n+2}}F_2(x_{2n+1})) + k^{2n} a_2 \\
&\leq kd(x_{2n}, x_{2n+1}) + k^{2n+1} \leq \dots \\
&\leq k^{2n+1} d(x_0, x_1) + (2n+1)k^{2n+1}.
\end{aligned}$$

It follows that for each $n = 1, 2, 3 \dots$

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) + nk^n.$$

Since $k < 1$, it follows from Cauchy's root test that $\sum nk^n$ is convergent and hence $\{x_n\}$ is a Cauchy sequence in X , then there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$.

$$\begin{aligned}
d(z, {}^{a(z)}F_2(z)) &\leq d(z, x_{2n+1}) + d(x_{2n+1}, {}^{a(z)}F_2(z)) \\
&\leq d(z, x_{2n+1}) + H({}^{a_{2n+1}}F_1(x_{2n}), {}^{a(z)}F_2(z)) \\
&\leq d(z, x_{2n+1}) + \frac{a_1 d(z, {}^{a(z)}F_2(z)) [1 + d(x_{2n}, {}^{a_{2n+1}}F_1(x_{2n}))d(x_{2n}, {}^{a_{2n+1}}F_1(x_{2n}))]^{\frac{1}{2}}}{1 + 2d(x_{2n}, z)} \\
&\quad + a_2 d(x_{2n}, z) \\
&\rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

Hence $z \in {}^{a(z)}F_2(z)$.

Similarly $z \in {}^{a(z)}F_1(z)$.

Therefore $z \in {}^{a(z)}F_1(z) \cap {}^{a(z)}F_2(z)$. \square

Corollary 2.7. Let (X, d) be a complete metric space and let F_1 and F_2 be fuzzy mappings from X to $\mathcal{F}(X)$ satisfying the condition (i) of Theorem 2.6 and

$$\begin{aligned}
H({}^{a(x)}F_1(x), {}^{a(y)}F_2(y)) &\leq \frac{a_1 d(y, {}^{a(y)}F_2(y)) [d(x, {}^{a(x)}F_1(x))]^{\frac{1}{2}}}{1 + 2d(x, y)} \\
&\quad + a_2 d(x, y),
\end{aligned}$$

where $a_1, a_2 > 0$ and $a_1 + a_2 < 1$. Then there exists $z \in X$ such that $z \in {}^{a(z)}F_1(z) \cap {}^{a(z)}F_2(z)$.

Example 2.8. Let $X = [0, 1]$. For $x, y \in X$, $d(x, y) = |x - y|$ and $\alpha, \beta \in (0, 1]$. Define $F_1, F_2 : X \rightarrow \mathcal{F}(X)$ by

For $x = 0$,

$$F_1(0)(z) = \begin{cases} 1, & z = 0 \\ \frac{1}{2}, & 0 < z \leq 1/50 \\ 0, & z > 1/50 \end{cases} \quad F_2(0)(z) = \begin{cases} 1, & z = 0 \\ 1/4, & 0 < z \leq 1/100 \\ 0, & z > 1/100 \end{cases}$$

For $x \neq 0$,

$$F_1(x)(z) = \begin{cases} \alpha, & 0 \leq z < x/15 \\ \frac{\alpha}{2}, & x/15 \leq z \leq x/2 \\ 0, & z > x/2 \end{cases} \quad F_2(x)(z) = \begin{cases} \beta, & 0 \leq z < x/8 \\ \frac{\beta}{4}, & x/8 \leq z \leq x/2 \\ 0, & z > x/2 \end{cases}$$

Here ${}^a F_1(0) = {}^a F_2(0) = \{0\}$ if $\alpha = 1$ and

$${}^{\frac{1}{2}}F_1(x) = {}^{\frac{1}{2}}F_2(x) = [0, x/2]$$

For $x = y$, $H({}^{a(x)}F_1(x), {}^{a(y)}F_2(y)) = 0$.

For $x \neq y$,

$$H({}^{z(x)}F_1(x), {}^{z(y)}F_2(y)) < \frac{1/5|y - y/2|[(1 + |x - x/2|)|x - x/2|]^{\frac{1}{2}}}{1 + 2|x - y|} + 3/4|x - y|$$

Therefore F and G satisfy all the conditions of [Theorem 2.6](#) for taking the values $a_1 = 1/5$ and $a_2 = 3/4$ and 0 is the common fixed point of F and G .

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